Reflectionless Sturm-Liouville Equations.

Robert Sims

Department of Mathematics
University of California at Davis
Davis CA 95616, USA
Email: rjsims@math.ucdavis.edu

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Abstract

We consider compactly supported perturbations of periodic Sturm-Liouville equations. In this context, one can use the Floquet solutions of the periodic background to define scattering coefficients. We prove that if the reflection coefficient is identically zero, then the operators corresponding to the periodic and perturbed equations, respectively, are unitarily equivalent. In some appendices, we also provide the proofs of several basic estimates, e.g. bounds and asymptotics for the relevant m-functions.

1 Introduction

Much can be understood about the properties of one dimensional Sturm-Liouville equations by analyzing the corresponding reflection and transmission coefficients. The purpose of this article is to present some results concerning the implications of trivial scattering, i.e., situations in which there is an absence of reflection.

Throughout our work, we will consider equations of the form

$$-(pu')' + qu = \lambda u \tag{1.1}$$

where $\lambda \in \mathbb{C}$ and the real-valued functions $\frac{1}{p}$ and q are locally integrable with $p > 0$ almost everywhere. As a first example, consider equation (1.1) in the case that the coefficients $1 - p$ and q are additionally assumed to have compact support; take $\text{supp}(1 - p) \subset [0, 1]$ and $\text{supp}(q) \subset [0, 1]$ for simplicity. For such equations, one defines the classical scattering coefficients by examining the Jost solution of (1.1) which satisfies

$$u(x,k) = \begin{cases} e^{ikx} & \text{for } x \leq 0, \\ a(k)e^{ikx} + b(k)e^{-ikx} & \text{for } x \geq 1. \end{cases} \tag{1.2}$$

for $0 < k$ with $\lambda = k^2$. The coefficients $a(k)$ and $b(k)$ describe the reflected and transmitted parts of an incoming plane wave, a solution of the free equation, and in the physics literature one often defines the transmission coefficient by $t(k) := \frac{1}{a(k)}$ and the reflection coefficient by $r(k) := \frac{b(k)}{a(k)}$. In

our work, we find it more useful to deal with the coefficients $a(k)$ and $b(k)$ directly, and we label them the transmission and reflection coefficient, respectively. We will be specifically interested in cases where the equation (1.1) is *reflectionless*, i.e., in the event that $b(k) = 0$ for all $k > 0$.

Let us briefly recall some of the known results. In the specific case mentioned above, in which both $1 - p$ and q are supported in $[0, 1]$, one can consider the equation

$$-(\tilde{p}u')' + \tilde{q}u = \lambda u \tag{1.3}$$

where $\tilde{p}$ and $\tilde{q}$ are the 1-periodic extensions of p and q, respectively, to $\mathbb{R}$. It is easy to see, e.g. Lemma 3.1 of [18], that if equation (1.1) is reflectionless, then the periodic operator corresponding to (1.3) has gapless spectrum equal to $[0, \infty)$. If one assumes in addition that $p \equiv 1$, then a result due to Borg [3, 4] for continuous q, later extended to integrable q by Hochstadt [12], may be used to prove that reflectionless, i.e., gapless spectrum for the periodic operator, implies $q \equiv 0$. Such results correspond to the well known fact that there are no compactly supported solitons.

If p is sufficiently smooth, e.g. p and p' are absolutely continuous, then one may apply a unitary, Liouville-Green transformation, as is done in [10], and see that equation (1.1) is equivalent to a Schrödinger equation of the form $-u'' + Qu = \lambda u$, where Q is a function involving both p and q. Based on these observations, it was proven in Theorem 3.3 of [18] that if $1 - p$ and q have support $[0, D]$ with p and p' absolutely continuous, then (1.1) is reflectionless if and only if

$$q = \frac{1}{16}\frac{(p')^2}{p} - \frac{1}{4}p''. \tag{1.4}$$

Using this equation, one can readily see that if $q \equiv 0$, $1 - p$ is compactly supported, p and p' are absolutely continuous, and equation (1.1) is reflectionless, then $p \equiv 1$. Moreover, one may establish that such a result remains true with no additional smoothness assumptions required on p. Specifically, Proposition 4.1 of [18] states that if $q \equiv 0$, $1-p$ is compactly supported, and (1.1) is reflectionless then $p \equiv 1$. This result was proven by examining the asymptotics of the m-function corresponding to (1.1). In partcular, it did not use known results for the Schrödinger equation due to the fact that the classical Liouville-Green operator is ill-defined when p is not smooth.

In this article, we will generalize some of these results to reflectionless equations whose scattering coefficients are defined with respect to a periodic background. More specifically, we consider 1-periodic, real-valued functions p_0 and q_0 for which both $\frac{1}{p_0}$ and q_0 are locally integrable with $p_0 > 0$ almost everywhere. It is well known, see e.g. [8, 6, 21], that the operator

$$H_0 = -\frac{d}{dx}p_0\frac{d}{dx} + q_0 \tag{1.5}$$

on $L^2(\mathbb{R})$ has spectrum which consists of a union of bands. For λ in a stability interval, there exist linearly independent solutions of

$$-(p_0u')' + q_0u = \lambda u, \tag{1.6}$$

which we label by $\phi_\pm(\cdot, \lambda)$ and refer to as the *Floquet solutions* corresponding to (1.6), see Section 2 for a further discussion. Let $f \geq 0$ and g be real-valued, integrable functions with compact support contained in $[0, \infty)$. Define perturbations of the periodic coefficients introduced above by setting $\frac{1}{p} := \frac{1}{p_0} + f$ and $q := q_0 + g$. The *periodic scattering coefficients* are defined in terms of the solution of

$$-(pu')' + qu = \lambda u \tag{1.7}$$

which satisfies

$$u_+(x,\lambda) = \begin{cases} \phi_+(x,\lambda) & \text{for } x \leq 0, \\ a_p(\lambda)\phi_+(x,\lambda) + b_p(\lambda)\phi_-(x,\lambda) & \text{for } x \geq D, \end{cases} \tag{1.8}$$

again, for λ in a stability interval. Here $D > 0$ is chosen as $\inf\{D' > 0 : \operatorname{supp}(f) \cup \operatorname{supp}(g) \subset [0, D']\}$. Comparing with (1.2), we see that the Floquet solutions play the role of the plane waves when the periodic background is non-trivial. Analogously to equation (1.5), we will denote by H the operator on $L^2(\mathbb{R})$ corresponding to the coefficients p and q. We say that equation (1.7) (the operator H) is *reflectionless* with respect to (1.6) (the operator H_0) if there exists a non-empty stability interval in the spectrum of H_0 on which $b_p(\lambda)$ is identically zero. We note that due to the analyticity of b_p, see Lemma 2.3 below, it is sufficient to assume that there exists a stability interval on which the zeros of b_p have an accumulation point.

We prove the following theorem.

Theorem 1.1. *If H is reflectionless with respect to H_0, then H is unitarily equivalent to H_0.*

Our proof follows from a result of Bennewitz [2] which establishes the existence of a more general Liouville Transform. Moreover, since this unitary map is explicit, we may also prove

Corollary 1.2. *Let p_0, q_0, f, and g be given as above. Suppose that $f \equiv 0$ and H is reflectionless with respect to H_0, then $g \equiv 0$.*

The paper is organized as follows. In Section 2, we introduce scattering coefficients defined with respect to a periodic background. Next, we prove Theorem 1.1 and Corollary 1.2 in Section 3. In the appendices that follow, we provide a proof of the technical estimates necessary to apply the inverse results found in [2]. The first appendix, Appendix A, establishes some bounds on the growth of solutions to Sturm-Liouville equations. A convergence result for the corresponding m-functions is given in Appendix B. Appendix C contains the main results which describe the asymptotics of the m-function and thereby the Weyl solution.

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2 Scattering at a Periodic Background

In this section, we recall some of the basic facts concerning periodic Sturm-Liouville equations. For a more detailed discussion of periodic problems, we refer the reader to [8]. We also note that many of the results stated below are proven, for Schrödinger equations, in [7].

2.1 The Floquet Solutions

Let p_0 and q_0 be 1-periodic, real valued functions for which both $\frac{1}{p_0}$ and q_0 are locally integrable and $p_0 > 0$ almost everywhere. Consider the self-adjoint operator on $L^2(\mathbb{R})$ given by

$$H_0 := -\frac{d}{dx}p_0\frac{d}{dx} + q_0. \tag{2.1}$$

The domain of this operator is the set of all $f \in L^2(\mathbb{R})$ for which both f and $p_0 f'$ are absolutely continuous and $H_0 f \in L^2(\mathbb{R})$. Since p_0 is not assumed to be smooth, in this generality we allow p_0

to be a step function for example, we note that the smoothness of $p_0 f'$ is not necessarily inherited by f'. For any $z \in \mathbb{C}$, let $u_N(\cdot, z)$ and $u_D(\cdot, z)$ denote the solutions of

$$-(p_0 u')' + q_0 u = zu, \tag{2.2}$$

satisfying

$$\begin{pmatrix} u_N(0,z) & u_D(0,z) \\ p_0 u_N'(0,z) & p_0 u_D'(0,z) \end{pmatrix} = I. \tag{2.3}$$

Take $g_0(z)$ to be the transfer matrix of (2.2) from $x = 0$ to $x = 1$, i.e., the matrix for which

$$\begin{pmatrix} u(1,z) \\ p_0 u'(1,z) \end{pmatrix} = g_0(z) \begin{pmatrix} u(0,z) \\ p_0 u'(0,z) \end{pmatrix}, \tag{2.4}$$

for any solution u of (2.2). Set $\rho_\pm(z)$ to be the eigenvalues of $g_0(z)$, i.e., the roots of $\rho^2 - D(Z)\rho + 1 = 0$, where $D(z) = \mathrm{Tr}[g_0(z)]$. The spectrum of H_0 consists of bands which are given by the sets of real numbers E for which $|D(E)| \leq 2$. A *stability interval* of H_0 is a maximal interval, (c, d), such that $|D(E)| < 2$ for every $E \in (c, d)$. It is on such an interval, and appropriate analytic extensions thereof, that one may define the Floquet solutions. We state these results as a lemma.

Let p_0 and q_0 be as above and fix a stability interval (c, d) in the spectrum of the operator H_0. Consider the open vertical strip in the complex plane containing (c, d), i.e.,

$$S_{(c,d)} := \{z \in \mathbb{C} : z = E + i\eta, \text{ where } c < E < d \text{ and } \eta \in \mathbb{R}\}. \tag{2.5}$$

Lemma 2.1. *Let p_0, q_0, and $S_{(c,d)}$ be taken as above.*
i) *As functions of z, the eigenvalues $\rho_\pm$ of g_0 may be chosen analytic in $S_{(c,d)}$ with, at most, algebraic singularities at the points $z = c$ and $z = d$.*
ii) *For the choices of $\rho_\pm$ taken in i) above, one may define eigenvectors $v_\pm$ of g_0, corresponding to $\rho_\pm$, that are analytic in $S_{(c,d)}$ with, at most, algebraic singularities at c and d.*

Using Lemma 2.1, one defines the Floquet solutions to be the solutions of (2.2) which satisfy the initial conditions

$$\begin{pmatrix} \phi_\pm(0,z) \\ p_0 \phi_\pm'(0,z) \end{pmatrix} = v_\pm(z), \tag{2.6}$$

for any $z \in S_{(c,d)}$. We state the properties of these solutions as a separate lemma.

Lemma 2.2. *Let p_0, q_0, and $S_{(c,d)}$ be as above, and define the Floquet solutions $\phi_\pm(\cdot, z)$ as in (2.6) above. We have that*
i) *For any fixed x, both the solutions $\phi_\pm(x, \cdot)$ and the corresponding derivatives $p_0 \phi_\pm'(x, \cdot)$ are analytic in $S_{(c,d)}$ with, at most, algebraic singularities at the points $z = c$ and $z = d$.*
ii) *For every fixed $z \in S_{(c,d)}$, the set $\{\phi_+(\cdot, z), \phi_-(\cdot, z)\}$ constitutes a basis for the solution space corresponding to (2.2).*
iii) *Upon labeling the eigenvalues $\rho_\pm$ appropriately, for any $E \in (c, d)$, one has that $\phi_\pm(\cdot, E + i\eta) \in L^2$ near $\pm\infty$ for $\eta > 0$ and similarly $\phi_\pm(\cdot, E + i\eta) \in L^2$ near $\mp\infty$ for $\eta < 0$.*

The proofs of both Lemma 2.1 and Lemma 2.2 are provided in Section 2.1 of [7] in the context of Schrödinger equations. Substituting the definitions given above, one may easily translate these proofs to the Sturm-Liouville equations we consider here.

2.2 Periodic Scattering Coefficients

Fix p_0, q_0, and $S_{(c,d)}$ as defined in the previous subsection. As in the introduction, let $f \geq 0$ and g be real-valued, integrable functions whose supports are contained in $[0, D]$. Define perturbations of p_0 and q_0 by the equations $\frac{1}{p} := \frac{1}{p_0} + f$ and $q := q_0 + g$, respectively. For any $z \in S_{(c,d)}$, let u_+ be the solution of

$$-(pu')' + qu = zu \tag{2.7}$$

satisfying

$$u_+(x,z) := \begin{cases} \phi_+(x,z) & \text{for } x \leq 0 \\ a_p(z)\phi_+(x,z) + b_p(z)\phi_-(x,z) & \text{for } x \geq D. \end{cases} \tag{2.8}$$

As indicated by Lemma 2.2 ii), the Floquet solutions are linearly independent for $z \in S_{(c,d)}$, and therefore, $a_p(z)$ and $b_p(z)$ are uniquely defined. In this setting, b_p and u_+ take on the role of a modified reflection coefficient and Jost solution, respectively, relative to the periodic background.

Lemma 2.3. *Let (c,d) be a stability interval of H_0. The scattering coefficients $a_p(\cdot)$ and $b_p(\cdot)$ are analytic for $z \in S_{(c,d)}$ with, at most, algebraic singularities at the points $z = c$ and $z = d$.*

Proof. Let $g_1(z)$ denote the transfer matrix corresponding to (2.7) from $x = 0$ to $x = D$, in analogy with (2.4). Clearly then,

$$\begin{pmatrix} u_+(D,z) \\ pu_+'(D,z) \end{pmatrix} = g_1(z) \begin{pmatrix} \phi_+(0,z) \\ p_0\phi_+'(0,z) \end{pmatrix}. \tag{2.9}$$

It is well known that these solutions and their derivatives are, at fixed x, analytic in z, and therefore, the entries of $g_1(z)$ are entire as functions of z. Using Lemma 2.2 i), we conclude that the left hand side of (2.9) is analytic in $S_{(c,d)}$ with, at most, algebraic singularities at the points $z = c$ and $z = d$. From (2.8), one sees that

$$\begin{pmatrix} a_p(z) \\ b_p(z) \end{pmatrix} = \begin{pmatrix} \phi_+(D,z) & \phi_-(D,z) \\ p\phi_+'(D,z) & p\phi_-'(D,z) \end{pmatrix}^{-1} \begin{pmatrix} u_+(D,z) \\ pu_+'(D,z) \end{pmatrix}, \tag{2.10}$$

by which this lemma follows from another application of Lemma 2.2. $\square$

In what follows, we will denote by H the self-adjoint operator in $L^2(\mathbb{R})$ corresponding to equation (2.7) in analogy to (2.1). We will say that equation (2.7) (the operator H) is *reflectionless* with respect to equation (2.2) (the operator H_0) if there is a stability interval (c,d) in the spectrum of H_0 for which $b_p(\lambda) = 0$ for all $\lambda \in (c,d)$; thus, $b_p(z) = 0$ for all $z \in S_{(c,d)}$ by Lemma 2.3 above.

3 Proofs of the Main Results

In this section, we will prove Theorem 1.1 and Corollary 1.2. Our proofs are based on a recent inverse result of Bennewitz [2] in which he constructs a more general Liouville transform; specifically, one that is applicable even in the case of non-smooth p. We will begin by describing his transformations and then verify that we may apply his inverse results.

3.1 Liouville Transforms

Bennewitz's results apply to half-line operators. To state his main theorem, let p and q be real-valued functions on $[0, \infty)$ with both $\frac{1}{p}$ and q in $L^1_{\text{loc}}(0, \infty)$. Consider the Sturm-Liouville equation

$$-(pu')' + qu = \lambda u, \tag{3.1}$$

subject to the boundary condition

$$u(0)\cos(\alpha) + pu'(0)\sin(\alpha) = 0, \tag{3.2}$$

for some $\alpha \in [0, \pi)$. Bennewitz's results are applicable under rather general assumptions. For the results we wish to present, we will assume more than is necessary as we indicate briefly below.
i) We assume that we are working on a half-line; the results also apply in the case that p and q are as above, yet defined on $[0, b)$ with $b < \infty$.
ii) We assume that $p > 0$ almost everywhere. Technically, one need only assume $\frac{1}{p} \in L^1_{\text{loc}}$.
iii) We will assume that equation (3.1) is limit-point at $+\infty$, see Chapter 9 of [6] for details.

Definition 3.1. *A unitary Liouville transform is a map $F : L^2(0, \infty) \to L^2(0, \infty)$ satisfying*

$$(Fv)(x) = s(x)v(t(x)), \tag{3.3}$$

where $s \in L^2_{loc}(0, \infty)$ is such that $s \neq 0$ almost everywhere, $t(x) = \int_0^x |s(y)|^2 dy$, and $\lim_{b\to\infty} t(b) = \infty$.

One may easily check that such a map is unitary and that F^{-1}, the inverse of F, is also a unitary Liouville transform.

Now, for $i = 1, 2$, let p_i and q_i be functions which satisfy the conditions stated above and let $\alpha_i \in [0, \pi)$. Denote by H_i the self-adjoint operator on $L^2(0, \infty)$ generated by equation (3.1) and (3.2) with coefficients p_i and q_i and boundary conditions $\alpha_i \in [0, \pi)$. We use the following result from [2]:

Theorem 3.2. *Suppose the operators H_1 and H_2 have the same spectral measure. Then there is a unitary Liouville transform F mapping H_2 to H_1; specifically, $FH_2 = H_1F$.*

Bennewitz explicitly constructs this unitary Liouville transformation in terms of the mappings $t_i : [0, \infty) \to [0, \infty)$ defined by

$$t_i(x) := \int_0^x \left(\frac{1}{p_i(y)}\right)^{1/2} dy. \tag{3.4}$$

He then defines $t : [0, \infty) \to [0, \infty)$ as $t(x) = t_2^{-1}(t_1(x))$ and subsequently, for almost every $x \in [0, \infty)$,

$$s(x) := \sqrt{t'(x)} = \left(\frac{p_2(t(x))}{p_1(x)}\right)^{1/4} > 0. \tag{3.5}$$

We note that although the spectral measures of H_1 and H_2 depend on the boundary conditions α_1 and α_2, respectively, the functions s and t, and therefore the Liouville transform F, do not.

3.2 Proofs

We may now provide the proofs of Theorem 1.1 and Corollary 1.2. The proof uses well known results concerning the m-function corresponding to the Sturm-Liouville equations we are considering. We refer the interested reader to Chapter 9 of [6] for a complete discussion.

Proof of Theorem 1.1 If H is reflectionless with respect to H_0, then there exists a stability interval (c, d) in the spectrum of H_0 for which $b_p(\lambda) = 0$ for all $\lambda \in (c, d)$. As b_p is analytic on $S_{(c,d)}$, we conclude that $b_p(z) = 0$ for all $z \in S_{(c,d)}$. Using (2.8), we see that for $\lambda \in (c, d)$ and $\eta > 0$ the modified Jost solution satisfies

$$u_+(x, \lambda + i\eta) = a_p(\lambda + i\eta)\phi_+(x, \lambda + i\eta), \tag{3.6}$$

for all $x \geq D$. By Lemma 2.2 iii), the Floquet solution $\phi_+(\cdot, \lambda + i\eta)$ is square integrable at $+\infty$. Thus, the modified Jost solution u_+ coincides, up to a complex multiple, with the Weyl solution. Appealing again to (2.8), it is clear that for $\lambda + i\eta \in S_{(c,d)}$ with $\eta > 0$, the m-function for the perturbed equation (1.7) satisfies

$$m(0, \lambda + i\eta) = \frac{pu_+'(0, \lambda + i\eta)}{u_+(0, \lambda + i\eta)} = \frac{p_0\phi_+'(0, \lambda + i\eta)}{\phi_+(0, \lambda + i\eta)} = m_0(0, \lambda + i\eta), \tag{3.7}$$

where m_0 is the corresponding m-function for the periodic equation (1.6). As the m-functions are analytic on the upper half plane, the equality in (3.7) holds throughout the upper half plane. From equality of the m-functions, we can conclude the equality of the spectral measures of the half-line operators, with Dirichlet boundary condition at $x = 0$, corresponding to equations (1.6) and (1.7), respectively. Applying Bennewitz's result Theorem 3.2, we find an explicit unitary equivalence of the Dirichlet operators on $[0, \infty)$. Since the coefficients of (1.6) and (1.7) are identical on $(-\infty, 0]$, this unitary transformation, which does not depend on the boundary condition at $x = 0$, can be extended by the identity to a unitray transformation corresponding to the whole line. We have proven the theorem. □

Proof of Corollary 1.2: Suppose $f \equiv 0$, i.e., $p \equiv p_0$ and H is reflectionless with respect to H_0. In this case, it is clear from (3.4) that the mappings $t_1 \equiv t_2$, for this application we take t_1 to be defined in terms of p_0 and t_2 in terms of p, and therefore $t(x) = x$. Using (3.5), we see that $s \equiv 1$, and therefore, the unitary Liouville transform F is the identity. From this, the equation $FH = H_0F$ implies that $q_0(x) = q(x)$, i.e. $g \equiv 0$. □

Appendix

The goal of the appendices that follow is to prove the key technical estimate, Lemma 2.4 of [2], which enables Bennewitz to prove Theorem 3.2. Such estimates were originally proven by Bennewitz in [1], see specifically Theorem 6.1 and Corollary 6.2, and they are applicable under rather general assumptions; for example, the coefficients of the basic Sturm-Liouville equation may be taken to be measures. Our approach is more pedestrian, in particular, we assume the coefficients are in $L^1_{\text{loc}}(\mathbb{R})$, but we hope our streamlined presentation is more easily accessible.

The heart of the matter is contained in Appendix C where we prove Theorem C.5, the analogue of Lemma 2.4 of [2]. The proof of Theorem C.5 uses a convergence result for m-functions which are defined with respect to a sequence of Sturm-Liouville equations whose coefficients converge in $L^1_{\mathrm{loc}}(\mathbb{R})$. In Appendix B, we prove Lemma B.1 and Theorem B.2 which demonstrate the desired convergence of the m-functions. These results are new. Moreover, as is discussed in the appendix of [13] for equations with coefficients in $L^2_{\mathrm{loc}}(\mathbb{R})$, they constitute a generalization of the applicability of Kotani Theory, see [14, 15, 16], to the Sturm-Liouville equations considered here. Lastly, all the results presented in these appendices rely heavily on certain basic solution estimates. The proofs of these results, which are simple generalizations of estimates well known in the context of Schrodinger equations, i.e., when $p \equiv 1$, are presented in Appendix A.

A A Priori Solution Estimates

In this first appendix, we provide several standard solution estimates which we will use frequently in the appendices that follow. Although results of this type are well-known, see e.g. [6, 17, 21], we include the proofs here for the convenience of the reader.

The basic Sturm-Liouville equation we consider is

$$-(pu')' + qu = 0, \tag{A.1}$$

where it is assumed throughout that $\frac{1}{p} \in L^1_{loc}(\mathbb{R})$, $p > 0$ almost everywhere, and $q \in L^1_{loc}(\mathbb{R})$ may be complex valued. In most of our applications, we have $q = q_0 - \lambda$ for some real valued, locally integrable function q_0 and $\lambda \in \mathbb{C}$ a constant. For any $f \in L^1_{loc}(\mathbb{R})$ and $I \subset \mathbb{R}$, a bounded interval, we will denote by

$$\|f\|_I := \int_I |f(t)|dt. \tag{A.2}$$

Lemma A.1. *Let u be a solution of (A.1). For any $x, y \in \mathbb{R}$, we have that*

$$|u(x)|^2 + |pu'(x)|^2 \leq \left(|u(y)|^2 + |pu'(y)|^2\right) \exp\left(\left\|\frac{1}{p} + q\right\|_I\right), \tag{A.3}$$

where the interval $I := [\min(x, y), \max(x, y)]$.

Proof. Setting $R(t) := |u(t)|^2 + |pu'(t)|^2$, one easily calculates that

$$|R'(t)| = \left|2\mathrm{Re}\left[\left(\frac{1}{p(t)} + q(t)\right) u(t)\overline{pu'(t)}\right]\right| \leq \left|\frac{1}{p(t)} + q(t)\right| R(t). \tag{A.4}$$

Thus, $|(\ln R(t))'| \leq |\frac{1}{p(t)} + q(t)|$, and the lemma is proven. ☐

Lemma A.2. *For $i = 1, 2$, let p_i and q_i be functions with $\frac{1}{p_i} \in L^1_{loc}(\mathbb{R})$, $p_i > 0$ almost everywhere, and $q_i \in L^1_{loc}(\mathbb{R})$. Suppose u_i are solutions of $-(p_i u_i')' + q_i u_i = 0$ which satisfy $u_1(y) = u_2(y)$ and $p_1 u_1'(y) = p_2 u_2'(y)$ for some $y \in \mathbb{R}$. Then, for any $x \in \mathbb{R}$ there exists a constant $C > 0$ for which*

$$\begin{aligned} &\left(|u_1(x) - u_2(x)|^2 + |p_1 u_1'(x) - p_2 u_2'(x)|^2\right)^{1/2} \\ &\leq C \left(|u_1(y)|^2 + |p_1 u_1'(y)|^2\right)^{1/2} \exp\left(\sum_{i=1}^{2} \left\|\frac{1}{p_i} + |q_i|\right\|_I\right), \end{aligned} \tag{A.5}$$

and one may take

$$C^2 = \left\| \frac{1}{p_1} - \frac{1}{p_2} \right\|_I^2 + \|q_1 - q_2\|_I^2. \tag{A.6}$$

Here $I := [\min(x,y), \max(x,y)]$.

Proof. Without loss of generality, we consider the case of $y \le x$. For $i = 1, 2$, define the vector

$$\vec{u}_i(t) := \begin{pmatrix} u_i(t) \\ p_i u_i'(t) \end{pmatrix}, \tag{A.7}$$

for any $t \in \mathbb{R}$. Using this notation, the solutions u_1 and u_2 clearly satisfy

$$\begin{aligned} \vec{u}_1(s) - \vec{u}_2(s) \quad &= \quad \int_y^s \begin{pmatrix} \left(\frac{1}{p_1(t)} - \frac{1}{p_2(t)}\right) p_1 u_1'(t) \\ (q_1(t) - q_2(t)) u_1(t) \end{pmatrix} dt + \\ &+ \quad \int_y^s \begin{pmatrix} 0 & \frac{1}{p_2(t)} \\ q_2(t) & 0 \end{pmatrix} (\vec{u}_1(t) - \vec{u}_2(t)) \; dt, \end{aligned} \tag{A.8}$$

for any $y \le s \le x$. With the usual vector norm $\| \cdot \|$, one may estimate that

$$\begin{aligned} \|\vec{u}_1(s) - \vec{u}_2(s)\| \; \le \; \max \left\{ \sup_{t \in I} |u_1(t)|, \sup_{t \in I} |p_1 u_1'(t)| \right\} C + \\ + \int_y^s \left(\frac{1}{p_2(t)} + |q_2(t)| \right) \|\vec{u}_1(t) - \vec{u}_2(t)\| \, dt. \end{aligned} \tag{A.9}$$

By Lemma A.1, we may conclude that for $\psi \in \{u_1, p_1 u_1'\}$,

$$\sup_{t \in I} |\psi(t)| \le \exp \left(\frac{1}{2} \left\| \frac{1}{p_1} + q_1 \right\|_I \right) \|\vec{u}_1(y)\|. \tag{A.10}$$

An application of Gronwall's lemma, see e.g. [20], to inequality (A.9) yields (A.5) as desired. □

In the next lemmas we will provide local estimates from below on the average growth of solutions to equation (A.1). For any function $f \in L^1_{\text{loc}}(\mathbb{R})$ and any compact interval $I \subset \mathbb{R}$, we will denote by

$$\|f\|_{I,\text{loc}+} \; := \; \sup_{x \in I} \int_x^{x+1} |f(t)| dt \; < \; \infty, \tag{A.11}$$

and

$$\|f\|_{I,\text{loc}-} \; := \; \inf_{x \in I} \int_x^{x+1} |f(t)| dt \; \ge \; 0. \tag{A.12}$$

Remark A.3. *Since we assume throughout that $p > 0$ and $\frac{1}{p} \in L^1_{\text{loc}}(\mathbb{R})$, for any compact (non-empty) interval $I \subset \mathbb{R}$, the function $P : I \to (0, \infty)$ defined by*

$$P(x) \; := \; \int_x^{x+1} \frac{1}{p(t)} dt, \tag{A.13}$$

is continuous; hence $\|\frac{1}{p}\|_{I,\text{loc}-} > 0$.

Lemma A.4. *Let p and q be functions with $\frac{1}{p} \in L^1_{\mathrm{loc}}(\mathbb{R})$, $p > 0$ almost everywhere, and $q \in L^1_{\mathrm{loc}}(\mathbb{R})$. For any (non-empty) interval $I = [a, b] \subset \mathbb{R}$, there exists $C > 0$ such that for all real valued solutions of $-(pu')' + qu = 0$ and any $x \in I$,*

$$\int_x^{x+2} |u(t)|^2 dt \geq C \left(|u(x)|^2 + |pu'(x)|^2 \right). \tag{A.14}$$

Proof. Fix $x \in I$ and set $\tilde{I} = [a, b+2]$. By Lemma A.1, there are constants $0 < C_1, C_2 < \infty$, depending only on $\|\frac{1}{p}\|_{\tilde{I},\mathrm{loc}+}$ and $\|q\|_{\tilde{I},\mathrm{loc}+}$, such that any solution of $-(pu')' + qu = 0$ satisfies

$$C_1 \left(|u(x)|^2 + |pu'(x)|^2 \right) \leq |u(t)|^2 + |pu'(t)|^2 \leq C_2 \left(|u(x)|^2 + |pu'(x)|^2 \right), \tag{A.15}$$

for all $t \in [x, x+2]$. With $C_3 := (C_1/2)^{1/2}$ and $C_4 := (2C_2)^{1/2}$, we also have that

$$C_3 \left(|u(x)| + |pu'(x)| \right) \leq |u(t)| + |pu'(t)| \leq C_4 \left(|u(x)| + |pu'(x)| \right). \tag{A.16}$$

We now claim that there exists an $x_0 \in [x + 1/2, x + 3/2]$ for which

$$|u(x_0)| \geq \frac{C_3}{4} \min \left(\left\| \frac{1}{p} \right\|_{\tilde{I},\mathrm{loc}-}, 1 \right) \left(|u(x)| + |pu'(x)| \right). \tag{A.17}$$

If this is not the case, then for all $t \in [x + 1/2, x + 3/2]$,

$$|u(t)| < \frac{C_3}{4} \min \left(\left\| \frac{1}{p} \right\|_{\tilde{I},\mathrm{loc}-}, 1 \right) \left(|u(x)| + |pu'(x)| \right). \tag{A.18}$$

Using (A.16), we conclude then that for all $t \in [x + 1/2, x + 3/2]$,

$$\begin{aligned} |pu'(t)| &\geq C_3 \left(|u(x)| + |pu'(x)| \right) - |u(t)| \qquad \text{(A.19)} \\ &> C_3 \left(|u(x)| + |pu'(x)| \right) \left(1 - \frac{\min(\|\frac{1}{p}\|_{\tilde{I},\mathrm{loc}-}, 1)}{4} \right) \\ &\geq \frac{C_3}{2} \left(|u(x)| + |pu'(x)| \right), \end{aligned}$$

i.e., pu' is strictly signed, and therefore

$$\begin{aligned} \frac{C_3}{2} \min \left(\left\| \frac{1}{p} \right\|_{\tilde{I},\mathrm{loc}-}, 1 \right) \left(|u(x)| + |pu'(x)| \right) &> |u(x + 3/2)| + |u(x + 1/2)| \\ &\geq \left| \int_{x+1/2}^{x+3/2} \frac{1}{p(t)} pu'(t) dt \right| \qquad \text{(A.20)} \\ &\geq \frac{C_3}{2} \left(|u(x)| + |pu'(x)| \right) \int_{x+1/2}^{x+3/2} \frac{1}{p(t)} dt, \end{aligned}$$

which is an obvious contradiction. We have proven (A.17).

Since the function $x \mapsto \int_a^x \frac{1}{p(t)} dt$ is continuous on $\tilde{I}$, it is uniformly continuous. Thus for $\varepsilon > 0$ defined by the equation $8C_4\varepsilon := C_3 \min\left(\left\|\frac{1}{p}\right\|_{\tilde{I},\mathrm{loc}-}, 1\right)$ there exists a $\delta > 0$ for which

$$\sup_{x\in I} \int_x^{x+\delta} \frac{1}{p(t)} dt \;\leq\; \varepsilon. \tag{A.21}$$

In this case, we conclude that for any $|t - x_0| \leq \delta$,

$$\begin{aligned} |u(t) - u(x_0)| \quad &\leq \quad C_4\left(|u(x)| + |pu'(x)|\right)\left|\int_{x_0}^t \frac{1}{p(s)} ds\right| \\ &\leq \quad \frac{C_3}{8}\left(|u(x)| + |pu'(x)|\right) \min\left(\left\|\frac{1}{p}\right\|_{\tilde{I},\mathrm{loc}-}, 1\right). \end{aligned} \tag{A.22}$$

From this observation, (A.14) follows. ∎

For certain applications, we will need a variant of Lemma A.4 which is true for complex valued solutions of (A.1); note the argument in (A.20) fails if the solution is not real valued.

Lemma A.5. *Let p and q be functions with $\frac{1}{p} \in L^1_{\mathrm{loc}}(\mathbb{R})$, $p > 0$ almost everywhere, and $q \in L^1_{\mathrm{loc}}(\mathbb{R})$. Suppose $u \neq 0$ is a solution of $-(pu')' + qu = 0$ which satisfies $Re[u(x)\overline{pu'(x)}] = 0$ for some $x \in R$. Then, there exists a constant C, depending only on the local L^1-norms of $\frac{1}{p}$ and q, for which*

$$\int_x^{x+2} |u(t)|^2 dt \geq C\left(|u(x)|^2 + |pu'(x)|^2\right). \tag{A.23}$$

Proof. It is enough to demonstrate (A.23) for solutions which additionally satisfy

$$|u(x)|^2 + |pu'(x)|^2 = 1, \tag{A.24}$$

since for an arbitrary solution of (A.1) one may define the normalized solution $\psi(t) := [|u(x)|^2 + |pu'(x)|^2]^{-1/2}\, u(t)$ which does satisfy (A.24). We continue as in the proof of the previous lemma.

For any solution of (A.1) which satisfies (A.24), there exists constants $0 < C_1 < C_2 < \infty$ for which

$$C_1 \;\leq\; |u(t)|^2 \;+\; |pu'(t)|^2 \;\leq\; C_2, \qquad \text{for all } t \in [x, x+2], \tag{A.25}$$

by Lemma A.1. Let $0 < a < 1$ be given; we will choose such an a below. We claim that there exists an $x_0 \in [x, x+1]$ for which

$$|u(x_0)| \;\geq\; aC_1. \tag{A.26}$$

If we establish the existence of an a and an x_0 for which (A.26) holds, then (A.23) is true as one may calculate that

$$p(t)\frac{d}{dt}|u(t)|^2 \;=\; 2\mathrm{Re}\left[\, u(t)\overline{pu'(t)}\,\right], \tag{A.27}$$

and therefore

$$\begin{aligned} \big|\, |u(t)|^2 \;-\; |u(x_0)|^2 \,\big| \quad &\leq \quad \int_{\min(t,x_0)}^{\max(t,x_0)} \frac{1}{p(s)}\left(|u(s)|^2 \;+\; |pu'(s)|\right) ds \\ &\leq \quad C_2 \int_{\min(t,x_0)}^{\max(t,x_0)} \frac{1}{p(s)}\, ds \;\leq\; \frac{aC_1}{2}, \end{aligned} \tag{A.28}$$

for t sufficiently small. As in Lemma A.4, this completes the proof.

To verify (A.26), suppose it is not the case. Then

$$|u(t)|^2 \;<\; aC_1, \qquad \text{for all } t \in [x, x+1], \tag{A.29}$$

and hence (A.25) implies that

$$|pu'(t)|^2 \;>\; (1-a)C_1, \qquad \text{for all } t \in [x, x+1], \tag{A.30}$$

as well. Using (A.27), one may further calculate that

$$\frac{d}{dt}\left(p(t)\frac{d}{dt}|u(t)|^2\right) \;=\; 2\left(\frac{|pu'(t)|^2}{p(t)} \;+\; \mathrm{Re}[q(t)]\,|u(t)|^2\right). \tag{A.31}$$

We can now estimate that

$$\int_x^{x+1} \frac{d}{dt}|u(t)|^2\,dt \;=\; |u(x+1)|^2 - |u(x)|^2 \;<\; aC_1. \tag{A.32}$$

Moreover,

$$\begin{aligned} \int_x^{x+1} \frac{d}{dt}|u(t)|^2\,dt \;&=\; \int_x^{x+1} \frac{1}{p(t)} \int_x^t \frac{d}{ds}\left(p(s)\frac{d}{ds}|u(s)|^2\right) ds\,dt \\ &=\; I_1 + I_2, \end{aligned} \tag{A.33}$$

where we have used (A.27), the boundary condition $\mathrm{Re}[u(x)\overline{pu'(x)}] = 0$,

$$I_1 \;=\; 2\int_x^{x+1} \frac{1}{p(t)} \int_x^t \frac{|pu'(s)|^2}{p(s)}\,ds\,dt, \tag{A.34}$$

and

$$I_2 \;=\; 2\int_x^{x+1} \frac{1}{p(t)} \int_x^t \mathrm{Re}[q(s)]\,|u(s)|^2\,ds\,dt. \tag{A.35}$$

With (A.30), it is clear that

$$I_1 \;\geq\; (1-a)C_1\left(\int_x^{x+1} \frac{1}{p(t)}dt\right)^2. \tag{A.36}$$

To bound I_2, we use (A.29) as follows

$$\begin{aligned} I_2 \;&\geq\; -2\int_x^{x+1} \frac{1}{p(t)} \int_x^t \mathrm{Re}[q(s)]_-\,|u(s)|^2\,ds\,dt \\ &\geq\; -2\,a\,C_1 \int_x^{x+1} \frac{1}{p(t)} \int_x^t \mathrm{Re}[q(s)]_-\,ds\,dt, \end{aligned} \tag{A.37}$$

where for a real valued function f, $f_-(t) := \max(-f(t), 0)$. Collecting our bounds from (A.32) to (A.37), we have proven that

$$aC_1 \;>\; aC_1\left[\frac{1-a}{a}\left(\int_x^{x+1}\frac{1}{p(t)}dt\right)^2 \;-\; 2\int_x^{x+1}\frac{1}{p(t)}\int_x^t \mathrm{Re}[q(s)]_-\,ds\,dt\right], \tag{A.38}$$

which is an obvious contradiction for a sufficiently small. We have established the claim (A.26), and thus the lemma is proven. □

Remark A.6. *One added difficulty in the proof of Lemma A.5 above, is the possibility that the solution u may vanish at x. If one knows the complex valued solution satisfies $|u(x)|^2 = 1$, in contrast to $Re[u(x)\overline{pu'(x)}] = 0$, then arguments as in (A.28) readily provide lower bounds of the type found in (A.23).*

B The m-function

The goal of this section is to prove Theorem B.2 below which concerns the compact uniform convergence of m-functions corresponding to a sequence of Sturm-Liouville equations whose coefficients converge in $L^1_{\rm loc}(\mathbb{R})$. For a more indepth discussion of m-function theory, we refer the reader to Chapter 9 of [6], and for convenience, we adopt the notation used therein.

Let $\{p_n\}_{n=0}^\infty$ and $\{q_n\}_{n=0}^\infty$ be sequences of real-valued functions which satisfy $p_n > 0$, $\frac{1}{p_n} \in L^1_{\rm loc}(\mathbb{R})$, and $q_n \in L^1_{\rm loc}(\mathbb{R})$ for all $n \geq 0$. Let $K \subset \mathbb{C}^+$, the complex upper half plane, be compact and consider solutions of

$$-(p_n u')' + q_n u = \lambda u \tag{B.1}$$

for $\lambda \in K$. For $n \geq 0$ and $x \in \mathbb{R}$, let $\phi_{n,x}$ and $\theta_{n,x}$ be the solutions of (B.1) which satisfy the boundary conditions

$$\begin{pmatrix} \phi_{n,x}(x,\lambda) & \theta_{n,x}(x,\lambda) \\ p\phi'_{n,x}(x,\lambda) & p\theta'_{n,x}(x,\lambda) \end{pmatrix} = I. \tag{B.2}$$

Given any $y > x$ denote by

$$D^y_{n,x,\lambda} := \left\{ m \in \mathbb{C}^+ : \int_x^y |\phi_{n,x}(t,\lambda) + m\theta_{n,x}(t,\lambda)|^2 dt \leq \frac{\mathrm{Im}[m]}{\mathrm{Im}[\lambda]} \right\} \tag{B.3}$$

the disc of radius $r^y_{n,x,\lambda}$ and center $c^y_{n,x,\lambda}$ in the upper half plane. In the limit as $y \to \infty$, the boundary of these discs is given by the solutions, $m_n(x,\lambda)$, of the equation

$$\int_x^\infty |\phi_{n,x}(t,\lambda) + m_n(x,\lambda)\theta_{n,x}(t,\lambda)|^2 dt = \frac{\mathrm{Im}[m_n(x,\lambda)]}{\mathrm{Im}[\lambda]}. \tag{B.4}$$

Lemma B.1. *Let $\{p_n\}_{n=0}^\infty$ and $\{q_n\}_{n=0}^\infty$ be sequences of real-valued functions which satisfy $p_n > 0$, $\frac{1}{p_n} \in L^1_{\rm loc}(\mathbb{R})$, and $q_n \in L^1_{\rm loc}(\mathbb{R})$ for all $n \geq 0$. Fix $I = [a,b] \subset \mathbb{R}$ and $K \subset \mathbb{C}^+$ compact. If $\frac{1}{p_n} \to \frac{1}{p_0}$ and $q_n \to q_0$ in $L^1_{\rm loc}(\mathbb{R})$, then the union of discs*

$$D := \bigcup_{\lambda \in K, n \geq 0, x \in I} D^{b+2}_{n,x,\lambda} \tag{B.5}$$

is a bounded subset of $\mathbb{C}^+$. Moreover, every solution $m_n(x,\lambda)$ of (B.4) satisfies

$$\mathrm{Im}[m_n(x,\lambda)] \geq C > 0, \tag{B.6}$$

where C may be choosen uniform in the parameters: $n \geq 0$, $x \in I$, and $\lambda \in K$.

Proof. We will prove that the union of discs D is bounded by deriving uniform estimates on the center and radius of each $D^{b+2}_{n,x,\lambda}$. Recall from [6] that

$$c^{b+2}_{n,x,\lambda} = -\frac{[\phi_{n,x}, \theta_{n,x}](b+2)}{2i\mathrm{Im}[\lambda] \cdot \int_x^{b+2} |\theta_{n,x}(t,\lambda)|^2 dt}, \tag{B.7}$$

and

$$r_{n,x,\lambda}^{b+2} \;=\; \frac{1}{2\mathrm{Im}[\lambda]\cdot\int_x^{b+2}|\theta_{n,x}(t,\lambda)|^2dt}, \tag{B.8}$$

where $[\phi_{n,x},\theta_{n,x}](y)=\phi_{n,x}(y)\overline{p_n\theta'_{n,x}}(y)-p_n\phi'_{n,x}(y)\overline{\theta_{n,x}}(y)$. With $\tilde{I}=[a,b{+}2]$, the L^1_{loc}-convergence of the coefficients of (B.1) implies that:

$$\max\left\{\sup_{n\geq 0}\left\|\frac{1}{p_n}\right\|_{\tilde{I},+},\quad \sup_{n\geq 0}\sup_{\lambda\in K}\|q_n-\lambda\|_{\tilde{I},+}\right\} \;<\; \infty, \tag{B.9}$$

$$\inf_{n\geq 0}\left\|\frac{1}{p_n}\right\|_{\tilde{I},-} \;>\; 0, \tag{B.10}$$

and for every $\varepsilon>0$, there exists $\delta>0$ for which

$$\sup_{n\geq 0}\sup_{x\in I}\int_x^{x+\delta}\frac{1}{p_n(t)}dt \;\leq\; \varepsilon. \tag{B.11}$$

Inserting (B.9), (B.10), and (B.11) into the proof of Lemma A.5, one estimates that

$$\begin{aligned}\int_x^{b+2}|\theta_{n,x}(t,\lambda)|^2dt \;&\geq\; \int_x^{x+2}|\theta_{n,x}(t,\lambda)|^2dt\\ &\geq\; C\left(|\theta_{n,x}(x,\lambda)|^2+|p_n\theta'_{n,x}(x,\lambda)|^2\right)\\ &=\; C,\end{aligned} \tag{B.12}$$

with a constant $C>0$ which is uniform in the parameters: $x\in I$, $n\geq 0$, and $\lambda\in K$. From this bound, it is clear that

$$|r_{n,x,\lambda}^{b+2}| \;\leq\; \inf_{\lambda\in K}\frac{(2C)^{-1}}{\mathrm{Im}[\lambda]}, \tag{B.13}$$

and

$$|c_{n,x,\lambda}^{b+2}| \;\leq\; \inf_{\lambda\in K}\frac{C'}{\mathrm{Im}[\lambda]}, \tag{B.14}$$

where we note that the constant C', appearing in (B.14), incorporates a repeated application of Lemma A.1 to bound the generalized Wronskian $[\phi_{n,x},\theta_{n,x}](b+2)$. This proves the boundedness of D.

Similarly, using (B.3) and Remark A.6, one sees that any point $m\in D_{n,x,\lambda}^{b+2}$ satisfies

$$\begin{aligned}\mathrm{Im}[m] \;&\geq\; \mathrm{Im}[\lambda]\int_x^{x+2}|\phi_{n,x}(t,\lambda)+m\theta_{n,x}(t,\lambda)|^2dt\\ &\geq\; \mathrm{Im}[\lambda]\,C\,\left(1+|m|^2\right)\\ &\geq\; C\,\inf_{\lambda\in K}\mathrm{Im}[\lambda],\end{aligned} \tag{B.15}$$

which proves (B.6). □

Theorem B.2. *Let $\{p_n\}_{n=0}^\infty$ and $\{q_n\}_{n=0}^\infty$ be sequences of real-valued functions which satisfy $p_n > 0$, $\frac{1}{p_n} \in L^1_{\text{loc}}(\mathbb{R})$, and $q_n \in L^1_{\text{loc}}(\mathbb{R})$ for all $n \geq 0$. Fix $I = [a,b] \subset \mathbb{R}$ and $K \subset \mathbb{C}^+$ compact. If $\frac{1}{p_n} \to \frac{1}{p_0}$ and $q_n \to q_0$ in $L^1_{\text{loc}}(\mathbb{R})$ and the equation*

$$-(p_0 u')' + q_0 u = \lambda u \tag{B.16}$$

is limit point at $+\infty$, then

$$m_n(x,\lambda) \to m_0(x,\lambda), \tag{B.17}$$

uniformly for $(x,\lambda) \in I \times K$.

Proof. To prove this theorem, we will first establish pointwise convergence. Fix $x_0 \in I$ and $\lambda_0 \in K$. As (B.16) is limit point at $+\infty$, the disc D^y_{0,x_0,λ_0} shrinks to a unique point in the limit $y \to \infty$. Thus, by taking y large, we can make the radius r^y_{0,x_0,λ_0} arbitrarily small. For y' large but fixed, Lemma A.2 implies that $r^{y'}_{n,x_0,\lambda_0} \to r^{y'}_{0,x_0,\lambda_0}$ as the functions θ_{n,x_0} all satisfy the same normalization. From this we conclude that for all sufficiently large n, the radii $r^{y'}_{n,x_0,\lambda_0}$ can be made arbitrarily small as well. Moreover, since $m_0(x_0,\lambda_0)$ is in the interior of $D^{y'}_{0,x_0,\lambda_0}$ for all $y' > x_0$, we see from (B.3) that

$$\int_{x_0}^{y'} |\phi_{0,x_0}(t,\lambda_0) + m_0(x_0,\lambda_0)\theta_{0,x_0}(t,\lambda_0)|^2 dt < \frac{\text{Im}[m_0(x_0,\lambda_0)]}{\text{Im}[\lambda_0]}. \tag{B.18}$$

Appealing again to Lemma A.2, we conclude that for n sufficiently large,

$$\int_{x_0}^{y'} |\phi_{n,x_0}(t,\lambda_0) + m_0(x_0,\lambda_0)\theta_{n,x_0}(t,\lambda_0)|^2 dt < \frac{\text{Im}[m_0(x_0,\lambda_0)]}{\text{Im}[\lambda_0]}, \tag{B.19}$$

as well. In other words, for large enough n, $m_0(x_0,\lambda_0) \in D^{y'}_{n,x_0,\lambda_0}$, and therefore, $|m_n(x_0,\lambda_0) - m_0(x_0,\lambda_0)| \leq 2 \cdot r^{y'}_{n,x_0,\lambda_0}$. As we have already argued, these radii become arbitrarily small, and therefore we have proven pointwise convergence. In fact, as the functions $m_n(x_0,\cdot)$ are analytic in $\mathbb{C}^+$, the uniform bounds proven in Lemma B.1 combined with Montel's Theorem, see e.g. [19], imply that the pointwise convergence is uniform for $\lambda \in K$.

The full result now follows from the Ricatti equation, i.e., the fact that

$$\partial_x m_n(x,\lambda) = q_n(x) - \lambda - \frac{1}{p_n(x)} m_n(x,\lambda)^2. \tag{B.20}$$

For any $a \leq t \leq b$, integration of (B.20) yields the following estimate:

$$\begin{aligned} |m_n(t,\lambda) - m_0(t,\lambda)| \,\leq\, & |m_n(a,\lambda) - m_0(a,\lambda)| + \int_a^t |q_n(s) - q_0(s)| ds + \\ & + \int_a^t \left| \frac{1}{p_n(s)} - \frac{1}{p_0(s)} \right| \cdot |m_0(s,\lambda)|^2 ds + \int_a^t \frac{1}{p_n(s)} \cdot |m_n(s,\lambda)^2 - m_0(s,\lambda)^2| ds. \end{aligned} \tag{B.21}$$

As each $m_n(s,\lambda) \in D^{b+2}_{n,s,\lambda}$, Lemma B.1 implies that there exists $M > 0$ for which $|m_n(s,\lambda)| \leq M$ uniformly in the parameters $n \geq 0$, $s \in I$, and $\lambda \in K$. Inserting this upper bound into (B.21)

implies that

$$|m_n(t,\lambda) - m_0(t,\lambda)| \le |m_n(a,\lambda) - m_0(a,\lambda)| + \|q_n - q_0\|_I + M^2 \cdot \left\| \frac{1}{p_n} - \frac{1}{p_0} \right\|_I + 2M \cdot \int_a^t \frac{1}{p_n(s)} \cdot |m_n(s,\lambda) - m_0(s,\lambda)| ds. \quad \text{(B.22)}$$

Applying Gronwall's Lemma, see e.g. [20], we find that for any $t \in I$,

$$|m_n(t,\lambda) - m_0(t,\lambda)| \le C(I,\lambda,n) \cdot \exp\left(2M \cdot \sup_{n \ge 0} \left\| \frac{1}{p_n} \right\|_I \right), \quad \text{(B.23)}$$

where

$$C(I,\lambda,n) := |m_n(a,\lambda) - m_0(a,\lambda)| + \|q_n - q_0\|_I + M^2 \cdot \left\| \frac{1}{p_n} - \frac{1}{p_0} \right\|_I. \quad \text{(B.24)}$$

The pointwise result we proved above demonstrates that $C(I,\lambda,n) \to 0$ uniformly for $\lambda \in K$. We have proven the theorem. □

C m-Asymptotics and Solution Estimates

We may now reproduce certain estimates found in [1]. We will consider the equation

$$-(pu')' + qu = \lambda u, \quad \text{(C.1)}$$

where p and q are real valued functions with $p > 0$ almost everywhere, both $\frac{1}{p}$ and q are locally integrable, and $\lambda \in \mathbb{C}$. We impose a further condition on the coefficients, namely that (C.1) is limit point at $+\infty$. The crux of Bennewitz's argument is a clever rescaling of the coefficients in (C.1) as the energy parameter λ varies along a ray in the complex plane. From this, he derives an asymptotic formula for the m-function and a related result for the Weyl solution.

C.1 The Scaling

Fix $x \in \mathbb{R}$ and denote by $P_x : [x,\infty) \to [0, \int_x^\infty \frac{1}{p(s)} ds)$ and $W_x : [x,\infty) \to [0,\infty)$ the functions defined by

$$P_x(t) := \int_x^t \frac{1}{p(s)} ds \quad \text{and} \quad W_x(t) := t - x. \quad \text{(C.2)}$$

As both P_x and W_x are continuous and strictly increasing, we may define their inverses P_x^{-1} and W_x^{-1}, respectively, each of which is also continuous and strictly increasing. Consider the function $\tilde{f}_x : (0, \int_x^\infty \frac{1}{p(s)} ds) \to (0,\infty)$ defined by

$$\tilde{f}_x(t) = \frac{1}{t\, W_x\left(P_x^{-1}(t)\right)} = \frac{1}{t\left(P_x^{-1}(t) - x\right)}. \quad \text{(C.3)}$$

Observe that $\tilde{f}_x$ is continuous and strictly decreasing with $\lim_{t\to 0^+} \tilde{f}_x(t) = \infty$ and $\tilde{f}_x(t) \to 0$ as $t \to \int_x^\infty \frac{1}{p(s)} ds$. In this case, we set $f_x := \tilde{f}_x^{-1}$.

Lemma C.1. *For fixed $x \in \mathbb{R}$, we have that*

$$\lim_{r\to\infty} f_x(r) = 0, \quad \textit{while} \quad \lim_{r\to\infty} r f_x(r) = \infty. \tag{C.4}$$

Moreover, for any pair of numbers (r,t) with $r > 0$ and $t > x$,

$$r\,P_x(t)\,W_x(t) = 1 \quad \textit{if and only if} \quad r\,f_x(r)\,W_x(t) = 1. \tag{C.5}$$

In particular, for such a pair (r,t), $f_x(r) = P_x(t)$.

Proof. Let $x \in \mathbb{R}$ be fixed. As $\tilde{f}_x$ is invertible, for each $r > 0$ there exists a unique $t_r > 0$ for which $r = \tilde{f}_x(t_r)$. Using the above observations, as $r \to \infty$, $t_r \to 0$ proving the first claim in (C.4). Direct substitution shows that

$$r f_x(r) = \frac{1}{W_x\left(P_x^{-1}(t_r)\right)}, \tag{C.6}$$

from which the later portion of (C.4) is clear.

Next, suppose that $r > 0$ and $t > x$ are choosen to satisfy the equation $r\,P_x(t)\,W_x(t) = 1$. In this case,

$$f_x(r) = f_x\left(\frac{1}{P_x(t)\cdot W_x\left(P_x^{-1}(P_x(t))\right)}\right) = P_x(t) = \frac{1}{rW_x(t)}. \tag{C.7}$$

Similarly, if $r\,f_x(r)\,W_x(t) = 1$, then $f_x(r) = \frac{1}{rW_x(t)}$ and therefore $r = \tilde{f}_x(\frac{1}{rW_x(t)})$. Rewriting things, one sees that $W_x(t) = W_x(P_x^{-1}(\frac{1}{rW_x(t)}))$, which implies that $t = P_x^{-1}(\frac{1}{rW_x(t)})$, and hence $r\,P_x(t)\,W_x(t) = 1$. We have proven the lemma. $\square$

C.2 m-function asymptotics

Fix $x \in \mathbb{R}$ and let ϕ_x and θ_x be the solutions of (C.1) which satisfy the boundary conditions

$$\begin{pmatrix} \phi_x(x,\lambda) & \theta_x(x,\lambda) \\ p\phi_x'(x,\lambda) & p\theta_x'(x,\lambda) \end{pmatrix} = I. \tag{C.8}$$

The following theorem is proven in [6]:

Theorem C.2. *Let ϕ_x and θ_x be the solutions of (C.1), corresponding to $\lambda \in \mathbb{C}^+$, which satisfy the boundary conditions given by (C.8). The linear combination $\psi_x = \phi_x + m\theta_x$ has the property that m is the m-function, $m(x,\lambda)$, of (C.1) if and only if $\lim_{t\to\infty}[\psi_x,\psi_x](t) = 0$, where*

$$[f,g](t) := f(t)\overline{pg'}(t) - pf'(t)\overline{g}(t) \tag{C.9}$$

is the modified Wronskian corresponding to (C.1).

This theorem is a direct consequence of the equation

$$[\psi_x,\psi_x](y) = 2i\mathrm{Im}[\lambda]\int_x^y |\psi_x|^2 dt - 2i\mathrm{Im}[m] \tag{C.10}$$

which is used to define the discs $D^y_{x,\lambda}$ introduced previously in (B.3); we refer to [6, 9] for the details.

We will now consider the properties of solutions of equation (C.1) as the energy parameter is varied along a ray in the complex upper half plane. Fix $\mu \in \mathbb{C}^+$, take $r > 0$, and suppose $u(t, r\mu)$ is a solution of (C.1) corresponding to $\lambda = r\mu$. For fixed $x \in \mathbb{R}$ set $s := \frac{1}{rf_x(r)}$ where f_x is as defined above. Denote by

$$q_r(t) \;:=\; \frac{q(st+x)}{r}, \quad \frac{1}{p_r(t)} \;:=\; \frac{s}{f_x(r)p(st+x)}, \tag{C.11}$$

and $u_r(t) := u(st + x, r\mu)$. A short calculation shows that

$$-\frac{d}{dt}\left(p_r \frac{d}{dt} u_r\right) + (q_r - \mu)u_r \;=\; \frac{1}{r}\left[-(pu')' + (q - r\mu)u\right], \tag{C.12}$$

and we see that if u solves (C.1), then u_r solves

$$-\frac{d}{dt}\left(p_r \frac{d}{dt} u_r\right) + q_r u_r = \mu u_r. \tag{C.13}$$

Take $\theta_r(t) := \frac{\theta_x(st+x,r\mu)}{f_x(r)}$ and $\phi_r(t) := \phi_x(st + x, r\mu)$. Observe that both θ_r and ϕ_r are solutions of the modified equation (C.13) which satisfy the boundary conditions

$$\begin{pmatrix} \phi_r(0) & \theta_r(0) \\ p_r\dot{\phi}_r(0) & p_r\dot{\theta}_r(0) \end{pmatrix} \;=\; I, \tag{C.14}$$

where we have used the notation $\dot{f} := \frac{d}{dt} f$.

Lemma C.3. *If $m(x, r\mu)$ is the m-function corresponding to (C.1) on $[x, \infty)$, then*

$$m_r(0, \mu) \;:=\; f_x(r)m(x, r\mu), \tag{C.15}$$

is the m-function for (C.13) on $[0, \infty)$.

Proof. Let $\psi_r(t) := \phi_r(t)+m_r(0, \mu)\theta_r(t)$, where $m_r(0, \mu)$ is as defined in (C.15). One easily verifies that

$$[\psi_r, \psi_r](t) = f_x(r)[\psi_x, \psi_x](ts + x), \tag{C.16}$$

where, as before, $\psi_x = \phi_x + m(x, r\mu)\theta_x$. Using Theorem C.2, if $m(x, r\mu)$ is the m-function corresponding to (C.1) with $\lambda = r\mu$, then (C.16) implies that $\lim_{t\to\infty}[\psi_r, \psi_r](t) = 0$, and therefore the lemma follows from another application of Theorem C.2. □

Theorem C.4. *Suppose p and q are real valued functions for which $p > 0$, $\frac{1}{p} \in L^1_{loc}(\mathbb{R})$, $q \in L^1_{loc}(\mathbb{R})$, and equation (C.1) is limit point at ∞. Assume in addition that x is a Lebesgue point of $\frac{1}{p}$. Then, as $r \to \infty$,*

$$m(x, r\mu) = i\sqrt{r\mu} \cdot \sqrt{p(x)} + o(\sqrt{r}), \tag{C.17}$$

where the square root above is the principle branch and the convergence is uniform for μ in compact subsets of $\mathbb{C}^+$.

Proof. Fix an arbitrary $c > 0$. One may calculate that

$$\int_0^c |q_r(t)|\,dt \;=\; \frac{1}{r}\int_0^c |q(st+x)|\,dt \;=\; f_x(r)\int_x^{cs+x} |q(y)|\,dy. \tag{C.18}$$

By Lemma C.1, both s and $f_x(r)$ go to zero as r goes to infinity, and so the same is true for the above integral. A similar result holds for $\frac{1}{p_r}$. To see this, set $\tilde{s} := W_x^{-1}(s)$. Clearly then,

$$r\, f_x(r)\, W_x(\tilde{s}) \;=\; r\, f_x(r)\, s \;=\; 1, \tag{C.19}$$

and from Lemma C.1 we may conclude that $f_x(r) = P_x(\tilde{s})$. In this case, one may calculate that

$$\int_0^c \frac{1}{p_r(t)}dt \;=\; \frac{s}{f_x(r)}\int_0^c \frac{1}{p(ts+x)}dt \;=\; \frac{W_x(\tilde{s})}{P_x(\tilde{s})}\,\frac{P_x(cs+x)}{cs}\,c. \tag{C.20}$$

Observe that as $r \to \infty$, $\tilde{s} \to x$. Thus, since x is a Lebegue point of $\frac{1}{p}$, the product of the ratios $\frac{W_x(\tilde{s})}{P_x(\tilde{s})}$ and $\frac{P_x(cs+x)}{cs}$ goes to one as $r \to \infty$. These calculations show that as $r \to \infty$, $q_r \to 0$ and $\frac{1}{p_r} \to 1$ in $L^1_{loc}(0, \infty)$. Using Theorem B.2, we may conclude that

$$\lim_{r\to\infty} m_r(0, \mu) = i\sqrt{\mu}, \tag{C.21}$$

which is the well-known m-function for the free equation.

Applying Lemma C.1 again, (C.19) implies not only that $f_x(r) = P_x(\tilde{s})$, but also $r\, P_x(\tilde{s})\, W_x(\tilde{s}) = 1$. From these equations it is easy to see that

$$\sqrt{r} f_x(r) \;=\; \sqrt{\frac{1}{P_x(\tilde{s})\, W_x(\tilde{s})}}\, P_x(\tilde{s}) \;=\; \sqrt{\frac{P_x(\tilde{s})}{W_x(\tilde{s})}}, \tag{C.22}$$

and thus

$$\lim_{r\to\infty} \sqrt{r} f_x(r) \;=\; \lim_{\tilde{s}\to x} \sqrt{\frac{P_x(\tilde{s})}{W_x(\tilde{s})}} \;=\; \sqrt{\frac{1}{p(x)}}, \tag{C.23}$$

where again we have used that x is a Lebesgue point of $\frac{1}{p}$. Clearly then,

$$\lim_{r\to\infty} \frac{m(x, r\mu)}{\sqrt{r}} \;=\; \lim_{r\to\infty} \frac{m_r(0,\mu)}{\sqrt{r} f_x(r)} = i\sqrt{\mu}\cdot\sqrt{p(x)}, \tag{C.24}$$

and we have proven (C.17). $\square$

C.3 The growth of solutions

For any $x \in \mathbb{R}$ and $\lambda \in \mathbb{C}^+$, the m-function may be written as

$$m(x, \lambda) = \frac{p\psi'(x, \lambda)}{\psi(x, \lambda)}, \tag{C.25}$$

where ψ is the Weyl solution corresponding to (C.1). Upon integration one finds that

$$\psi(x, \lambda) \;=\; \psi(0, \lambda) \;+\; \int_0^x \frac{1}{p(t)} m(t, \lambda)\psi(t, \lambda)dt, \tag{C.26}$$

and therefore,

$$\psi(x,\lambda) \;=\; \psi(0,\lambda)\,\exp\left(\int_0^x \frac{1}{p(t)} m(t,\lambda)dt\right). \tag{C.27}$$

We may now state

Theorem C.5. *Let $I := [0,b]$ and $K \subset \mathbb{C}^+$ compact be fixed. For any $\mu \in K$ and $r > 0$, let ψ be the Weyl solution of (C.1) corresponding to $\lambda = r\mu$. One has that*

$$\lim_{r\to\infty} \frac{1}{i\sqrt{r}} \ln\left[\frac{\psi(x,r\mu)}{\psi(0,r\mu)}\right] \;=\; \int_0^x \sqrt{\frac{\mu}{p(t)}}dt, \tag{C.28}$$

where the convergence is uniform for $(x,\lambda) \in I \times K$.

Formally, equation (C.28) follows readily by combining (C.17) and (C.27) above. Justifying the use of dominated convergence, however, requires a bit of work. To prove this theorem, we use a lemma, due to Hardy, concerning maximal functions.

Let μ_1 and μ_2 are two non-negative measures on $\mathbb{R}$ which are both absolutely continuous with respect to Lebesgue measure. Fix an open interval $I \subset \mathbb{R}$, and define for any $t \in I$

$$\mu(t) := \sup_{\substack{x,y\in I:\\ x<t<y}} \frac{\mu_1[(x,y)]}{\mu_2[(x,y)]}. \tag{C.29}$$

Lemma C.6. *Suppose $\mu_1[I] < \infty$, then*

$$\mu_2[\{t \in I : \mu(t) > s > 0\}] \le \frac{4\mu_1[I]}{s}. \tag{C.30}$$

A nice proof of this lemma appears in the appendix of [1].
Proof of Theorem C.5
Rewriting (C.27) yields

$$\frac{1}{i\sqrt{r}} \ln\left[\frac{\psi(x,r\mu)}{\psi(0,r\mu)}\right] \;=\; \int_0^x \frac{1}{p(t)}\frac{m(t,r\mu)}{i\sqrt{r}}\,dt \;=\; \int_0^x \frac{1}{p(t)}\frac{m_r(0,\mu)}{i\sqrt{r}f_t(r)}\,dt, \tag{C.31}$$

where, for the last equality above, we have used (C.15) and the quantities defined in the previous subsections. Define the following non-negative measure on $\mathbb{R}$

$$\mu_2[(a,b)] \;:=\; \int_a^b \frac{1}{p(t)}dt, \tag{C.32}$$

and note that (C.31) can be rewritten as

$$\frac{1}{i\sqrt{r}} \ln\left[\frac{\psi(x,r\mu)}{\psi(0,r\mu)}\right] \;=\; \int_0^x \frac{m_r(0,\mu)}{i\sqrt{r}f_t(r)}\,d\mu_2(t). \tag{C.33}$$

Using (C.22) and the fact that the convergence in (C.21) is uniform with respect to $\mu \in K$, as proven in Theorem B.2, we see that there exists $C > 0$ for which

$$\left|\frac{m_r(0,\mu)}{i\sqrt{r}f_t(r)}\right| \;\le\; C\sqrt{\frac{W_t(\tilde{s})}{P_t(\tilde{s})}}, \tag{C.34}$$

if r sufficiently large. Here $C = C(K)$. Now set

$$g(t) := \sup_{y \in (t,2b)} \frac{W_t(y)}{P_t(y)}. \tag{C.35}$$

We now claim that for r sufficiently large,

$$\left| \frac{m_r(0,\mu)}{i\sqrt{r} f_t(r)} \right| \leq C \sqrt{g(t)}. \tag{C.36}$$

Since I is compact and the function $P(y) := \int_0^y \frac{1}{p(t)}\, dt$ is continuous, the number $P_- := \inf_{y \in I} P(y+b) - P(y)$ is strictly positive. If r is chosen such that $rP_- b \geq 1$, then, as in the proof of Theorem C.4, with $s = \frac{1}{r f_t(r)}$ and $\tilde{s} = W_t^{-1}(s)$ we have that $rP_t(\tilde{s})W_t(\tilde{s}) = 1$. Thus,

$$\int_t^{\tilde{s}} \frac{1}{p(y)}\, dy\, (\tilde{s} - t) \;=\; P_t(\tilde{s})W_t(\tilde{s}) \;= \frac{1}{r} \;\leq\; P_- b, \tag{C.37}$$

from which it is clear that $\tilde{s} \leq b + t \leq 2b$. This proves (C.36).

We are now ready to apply Lemma C.6. Let $\tilde{I} = (0, 2b)$ be the open interval under consideration, let μ_1 be defined by $\mu_1[(a,b)] = b - a$, and let μ_2 be as defined in (C.32). Clearly, $\mu(t) \geq g(t)$, i.e. the two-sided maximal function μ dominates the one-sided maximal function g. From this inequality, the following set containment is obvious: $A_s := \{t \in I : g(t) > s > 0\} \subset \tilde{A}_s := \{t \in \tilde{I} : g(t) > s > 0\} \subset \{t \in \tilde{I} : \mu(t) > s > 0\}$, and therefore,

$$\mu_2[A_s] \;\leq\; \mu_2[\tilde{A}_s] \;\leq\; \mu_2[\{t \in \tilde{I} : \mu(t) > s > 0\}] \leq \frac{8b}{s}. \tag{C.38}$$

Hence, for any $x \in I$ one may estimate

$$\begin{aligned} \int_0^x \sqrt{g(t)} d\mu_2(t) &\leq \frac{1}{2}\int_0^1 s^{-1/2}\mu_2[A_s]ds + \frac{1}{2}\int_1^\infty s^{-1/2}\mu_2[A_s]ds \\ &\leq \mu_2[I] + 8b. \end{aligned} \tag{C.39}$$

Using the bounds in (C.36) and (C.39), we are justified in applying dominated convergence to (C.33), and the theorem is proven. □

References

[1] C. Bennewitz, *Spectral asymptotics for Sturm-Liouville equations,* Proc. London Math. Soc.(3) **59** (1989), 294–338.

[2] C. Bennewitz, *A Paley-Wiener theorem with applications to inverse spectral theory. Advances in differential equations and mathematical physics (Birmingham, AL, 2002), 21–31, Contemp. Math.*, **327**, Amer. Math. Soc., Providence, RI, 2003.

[3] G. Borg, Eine Umkehrung der Sturm-Liouvilleschen Eigenwertaufgabe. *Acta Math.* **78**, 1 – 96 (1946)

[4] G. Borg, Uniqueness theorems in the spectral theory of $y'' + (\lambda - q(x))y = 0$. *Proc. 11th Scandinavian Congress of Mathematicians*, Johan Grundt Tanums Forlag, Oslo (1952) 276–287

[5] D. Brown, personal communication (2001)

[6] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York, 1955.

[7] D. Damanik, R. Sims, and G. Stolz, Localization for one-dimensional, continuum, Bernoulli-Anderson models. *Duke Math. J.* **114** No. 1 (2002) 59–100

[8] M. S.P. Eastham, *The Spectral Theory of Periodic Differential Equations.* Texts in Math., Scottish Acad. Press, Edinburgh, 1973.

[9] M. Eastham and H. Kalf, *Schrödinger-type operators with continuous spectra. Research Notes in Mathematics.* **65**, Pitman (Advanced Publishing Program), Boston, Mass.-London, 1982.

[10] W. Everitt, On a property of the m-coefficient of a second-order linear differential equation. *J. London Math. Soc.* (2) **4** (1972) 443–457.

[11] P. Hartman and A. Wintner, *A criterion for the non-degeneracy of the wave equation,* Amer. J. Math. **71** (1949), 206–213.

[12] H. Hochstadt, On the Determination of a Hill's Equation from its Spectrum. *Arch. Rational Mech. Anal.* **19**, 353 – 362 (1965)

[13] W. Kirsch, S. Kotani, and B. Simon, Absence of absolutely continuous spectrum for some one-dimensional random but deterministic Schrodinger operators. *Ann. Inst. H. Poincar Phys. Thor.* **42** no. 4, 383–406 (1985)

[14] S. Kotani, Ljapunov indices determine absolutely continuous spectra of stationary random one-dimensional Schrdinger operators. *Stochastic analysis* (Katata/Kyoto, 1982), 225–247, North-Holland Math. Library, **32**, North-Holland, Amsterdam, 1984.

[15] S. Kotani, Support theorems for random Schrodinger operators. *Comm. Math. Phys.* **97** 443–452 (1985)

[16] S. Kotani, One-dimensional random Schrdinger operators and Herglotz functions. *Probabilistic methods in mathematical physics* (Katata/Kyoto, 1985), 219–250, Academic Press, Boston, MA, 1987.

[17] L., Pastur and A. Figotin, *Spectra of random and almost-periodic operators.* Fundamental Principles of Mathematical Sciences, 297, Springer-Verlag, Berlin, 1992.

[18] R. Sims and G. Stolz, Localization in one dimensional random media: A scattering theoretic approach. *Commun. Math. Phys.* **213** (2000) 575–597.

[19] E. Stein and R. Shakarchi, *Complex Analysis.* Princeton Lectures in Analysis II. Princeton University Press, Princeton-Oxford, (2003)

[20] W. Walter, *Ordinary Differential Equations*, vol. 182, Springer, New York, (1998)

[21] J. Weidmann, *Spectral Theory of Ordinary Differential Operators*, vol. 1258, Springer-Verlag, New York, (1987)

[22] J. Weidmann *Linear Operators in a Hilbert Space*, vol. 68 *Graduate Texts in Mathematics* Springer-Verlag, New York, (1980)

[23] J. Weidmann, *Lineare Operatoren in Hilberträumen. Teil 1. (German) [Linear operators in Hilbert spaces. Part 1]* Grundlagen. [Foundations] Mathematische Leitfäden. [Mathematical Textbooks] B. G. Teubner, Stuttgart, (2000)

[24] J. Weidmann, *Lineare Operatoren in Hilberträumen. Teil 2. (German) [Linear operators in Hilbert spaces. Part 2]* Anwendungen. B. G. Teubner, Stuttgart, (2003)